On weakly Radon-Nikodým compact spaces

Winter School in Abstract Analysis

Gonzalo Martínez Cervantes

University of Murcia, Spain

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- *A set $S \subset X$ is said to be* **weakly precompact** *if every sequence in S has a weakly Cauchy subsequence.*
- *X is* **weakly precompactly generated** *(WPG) if there exists a weakly precompact set $S \subset X$ such that $\overline{\operatorname{span}} S = X$.*

... By analogy with the well-known class of weakly compactly generated Banach spaces, one may call a Banach space such as X above a weakly precompactly generated (or WPG) space. The above example shows that WPG spaces exhibit certain pathologies that do not occur for WCG spaces, and indeed do not occur for the interesting 'WKA spaces' of Talagrand. The present author would be interested to know whether WPG spaces have any of the good properties of these other classes, and whether there is a nice characterization of those compact spaces T for which $\mathcal{C}(T)$ is WPG. One obvious question is whether every such space T contains a nontrivial convergent sequence.

R. Haydon, 1980

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- WLOG, $K \subset B_{X^*}$ with X not containing ℓ_1.
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- $T^* : X^* \to Y^*$ restricted to B_{X^*} is an embedding from B_{X^*} into the dual of a Banach space not containing ℓ_1.

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Theorem (H. Rosenthal, 1974)

There exists a non-WPG Banach space X such that B_{X^*} is WRN.

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Remark

A compact space K is WRN if and only if there exists a set Γ such that $K \hookrightarrow [0, 1]^{\Gamma}$ and for every $p < q$, the family of pairs $A_{\alpha}^0 = \{x \in K : x_{\alpha} < p\}$, $A_{\alpha}^1 = \{x \in K : x_{\alpha} > q\}$ with $\alpha \in \Gamma$ does not contain independent sequences.

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Lemma

Every WRN compact space is QWRN.

Definition

A compact space $K \hookrightarrow [0,1]^\Gamma$ is **quasi-WRN** *(QWRN for short) if for every $\varepsilon > 0$ there exists a decomposition $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n^\varepsilon$ such that for every $p < q$ with $q - p > \varepsilon$, the family of pairs $A_\alpha^0 = \{x \in K : x_\alpha < p\}$, $A_\alpha^1 = \{x \in K : x_\alpha > q\}$ with $\alpha \in \Gamma_n^\varepsilon$ does not contain independent sequences.*

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- Let $\mathcal{F} = \{f_\alpha\}_{\alpha\in\Gamma} \subset \mathcal{C}(K)$, where $f_\alpha(x) = \frac{x_\alpha}{n}$ for every $n \in \mathbb{N}$, $\alpha \in \Gamma_n$ and $x \in K$.

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- Then, $\mathcal{F}$ separates the points of K and it does not contain an independent sequence of functions. □

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Theorem (A. Avilés and P. Koszmider, 2011)

There exists a zero-dimensional RN compact space $\mathbb{L}_0$ *and a continuous surjection* $\pi : \mathbb{L}_0 \to \mathbb{L}_1$ *such that* $\mathbb{L}_1$ *is not RN.*

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Corollary

There is a QRN compact space which is not WRN.

Theorem

Every WRN compact space is in the class MS.

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Proof.

- K WRN $\Rightarrow \mathcal{C}(K)$ is WPG $\Rightarrow$ there exists a weakly precompact set $\mathcal{F}$ such that $\overline{\operatorname{span}}\,\mathcal{F} = \mathcal{C}(K)$.

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- K WRN $\Rightarrow \mathcal{C}(K)$ is WPG $\Rightarrow$ there exists a weakly precompact set $\mathcal{F}$ such that $\overline{\text{span}}\,\mathcal{F} = \mathcal{C}(K)$.
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- Therefore, $T(\mathcal{F})$ is a relatively $\|\cdot\|$-compact space with $\overline{\text{span}}\,T(\mathcal{F}) = L^1(\mu)$.
- In particular, $T(\mathcal{F})$ and $L^1(\mu)$ are separable $\Rightarrow$ μ is separable.

$\square$

- Every linearly ordered compact space is WRN (E. Glasner, M. Megrelishvili)

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Theorem (Stegall/J. Orihuela-W. Schachermeyer-M. Valdivia, 1991)

A compact space K is Eberlein if and only if it is Corson and RN.

$$\begin{array}{ccccccc}
 & & \text{QRN} & \longrightarrow & \text{QWRN} & & \\
 & & \uparrow & & \uparrow & & \\
 & & \text{RN} & \longrightarrow & \text{WRN} & \longrightarrow & \text{MS} \\
\text{Eberlein} & \longleftrightarrow & + & & & & \\
 & & \text{Corson} & & & &
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Theorem (V. Farmaki, 1985)

A compact space $K \subset \Sigma(\Gamma)$ is Eberlein if and only if for every $\varepsilon > 0$ there exists a decomposition $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n^\varepsilon$ such that for every $x \in K$ and every $n \in \mathbb{N}$, the set $\{\gamma \in \Gamma_n^\varepsilon : |x_\gamma| > \varepsilon\}$ is finite.

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The Talagrand's compact $T \subset \{0,1\}^{\omega^\omega}$ consisting of all functions 1_A with $A \subset \omega^\omega$ for which there exist $n \in \mathbb{N}$ and $s \in \omega^n$ such that

$$x|_n = y|_n = s \text{ but } x|_{n+1} \neq y|_{n+1} \text{ for every } x, y \in A \text{ with } x \neq y$$

is an example of a Corson compact that is not Eberlein.

Theorem (A. D. Arvanitakis, 2002)

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Lemma

Let $K \subset \Sigma(\Gamma)$ be a solid Corson compact space, where solid means that for every $x \in K$ and every $A \subset \Gamma$ finite, $x 1_A \in K$.

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Lemma

Let $K \subset \Sigma(\Gamma)$ be a solid Corson compact space, where solid means that for every $x \in K$ and every $A \subset \Gamma$ finite, $x1_A \in K$. Then, K is QWRN if and only if it is Eberlein.

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Corollary

The Talagrand's compact is solid, Corson and not Eberlein. Therefore, the Talagrand's compact is not QWRN.

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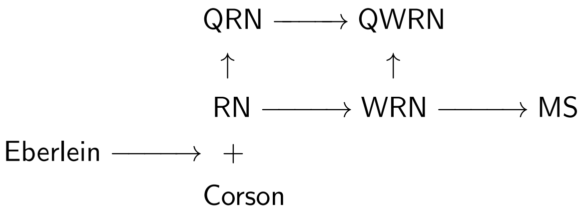
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There exists a WRN compact space which is not sequentially compact.

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Proof.

- Let $\mathcal{R}$ be a maximal family of subsets of $\mathbb{N}$ with respect to the condition that for $R, S \in \mathcal{R}$ at least one of the sets $R \cap S$, $R \cap S^c$ or $R^c \cap S$ is finite.

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- Take $X = \overline{\text{span}}\, C \subset \ell_\infty$. Since X is WPG, $K = B_{X^*}$ is WRN.
- The sequence $(e_n^*)_{n=1}^\infty$ does not have convergent subsequences in K.

□

Question (R. Haydon, 1980)

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- Efimov spaces exist under various set-theoretic assumptions.
- It is unknown whether a positive answer is consistent with ZFC.

Definition

- *Set an ordinal* $\epsilon > 0$*. An* **inverse system** *is a family* $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \epsilon \rangle$ *of compact spaces* K_α *and continuous functions* $f_{\alpha,\beta} : K_\beta \to K_\alpha$ *such that* $f_{\alpha,\gamma} \circ f_{\gamma,\beta} = f_{\alpha,\beta}$ *for any* $\alpha < \gamma < \beta$*.*

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- *If ϵ is a limit ordinal, then* **the limit of the inverse system** *is the subspace of $\prod_{\alpha<\varepsilon} K_\alpha$ consisting of those points $(x_\alpha)_{\alpha<\epsilon}$ that satisfy $f_{\alpha,\beta}(x_\beta) = x_\alpha$ for every $\alpha < \beta$.*

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- *$\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \epsilon \rangle$ is said to be* **continuous** *if for every limit ordinal $\gamma < \epsilon$, K_γ is the limit of the inverse system $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \gamma \rangle$.*

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- $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \epsilon \rangle$ *is said to be* **continuous** *if for every limit ordinal* $\gamma < \epsilon$, K_γ *is the limit of the inverse system* $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \gamma \rangle$.
- $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \epsilon \rangle$ *is said to be* **based on simple extensions** *if for every* $\alpha < \epsilon$ *there exists a point* $x_\alpha \in K_\alpha$ *such that* $|f^{-1}_{\alpha,\alpha+1}(x)| = 1$ *if* $x \neq x_\alpha$ *and* $|f^{-1}_{\alpha,\alpha+1}(x_\alpha)| = 2$.

Theorem (Koppelberg)

If $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \epsilon \rangle$ *is a continuous inverse system based on simple extensions with limit* K*, then* K *does not map onto* $[0,1]^{\omega_1}$ *unless* K_0 *does.*

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Let $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \omega_1\rangle$ be a continuous inverse system based on simple extensions with limit K. If $K_0 = 2^\omega$, then K is in the class MS.

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Let $\langle f_{\alpha,\beta}, K_\alpha : \alpha < \beta < \omega_1 \rangle$ be a continuous inverse system based on simple extensions with $K_0 = 2^\omega$ and with limit K. Is K WRN?

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